Surface Integrals

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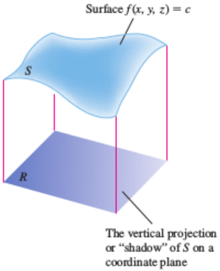
Overview : Surface Area and Surface Integrals

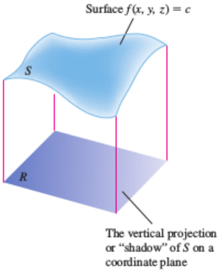
You know how to integrate a function over a flat region in a plane (double integral).

If the function is defined over a curved surface, we now discuss how to calculate its integral.

The trick to evaluating one of these so-called surface integrals is to rewrite it as a double integral over a region in a coordinate plane beneath the surface.



The Definition of Surface Area

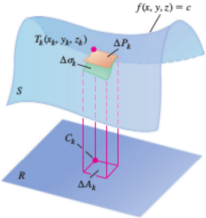


The figure shows a surface S lying above its "shadow" region R in a plane beneath it. The surface is defined by the equation $f(x, y, x) = c$.

The Definition of Surface Area

If the surface is **smooth** (∇f is continuous and never vanishes on S), we can define and calculate its area as a double integral over R.

The first step in defining the area of S is to partition the region R into small rectangles ΔA_k of the kind we would use if we were defining an integral over R.



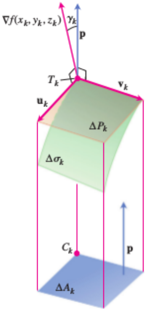
The Definition of Surface Area

Directly above each ΔA_k lies a patch of surface $\Delta\sigma_k$ that we may approximate with a portion ΔP_k of the tangent plane.

To be specific, we suppose that ΔP_k is a portion of the plane that is tangent to the surface at the point $T_k(x_k, y_k, z_k)$ directly above the back corner C_k of ΔA_k.

If the tangent plane is parallel to R, then ΔP_k will be congruent to ΔA_k. Otherwise, it will be a parallelogram whose area is somewhat larger than the area of ΔA_k.

The Definition of Surface Area



The figure gives a magnified view of $\Delta\sigma_k$ and ΔP_k, showing the gradient vector $\nabla f(x_k.y_k, z_k)$ at T_k and a unit vector p that is normal to R. The figure also show the angle γ_k between ∇f and p.

The Definition of Surface Area

The other vectors in the picture $\mathrm{u}_k \times \mathrm{v}_k$ and ∇f are normal to the tangent plane.

We now need the fact from advanced vector geometry that $|(\mathrm{u}_k \times \mathrm{v}_k.\mathrm{p})|$ is the area of the projection of the parallelogram determined by u_k and v_k onto any plane whose normal is p.

In our case, this translates into the statement

$$|(\mathrm{u}_k \times \mathrm{v}_k.\mathrm{p})| = \Delta A_k.$$

The Definition of Surface Area

Now, $|\mathsf{u}_k \times \mathsf{v}_k|$ itself is the area ΔP_k (standard fact about cross products). Hence

$$|\mathsf{u}_k \times \mathsf{v}_k| \;\; |\mathsf{p}| \;\; |\cos\text{(angle between } \mathsf{u}_k \times \mathsf{v}_k \text{ and } \mathsf{p})| = \Delta A_k.$$

$$\begin{aligned} \Delta P_k \; |\cos\gamma_k| &= \Delta A_k \\ \Delta P_k &= \frac{\Delta A_k}{|\cos\gamma_k|} \end{aligned}$$

provided $\cos\gamma_k \neq 0$.

We will have $\cos\gamma_k \neq 0$ as long as ∇f is not parallel to the ground plane and $\nabla f . \mathsf{p} \neq 0$.

The Definition of Surface Area

Since the patches ΔP_k approximate the surface patches $\Delta\sigma_k$ that fit together to make S, the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos\gamma_k|}$$

looks like an approximation of what we might like to call the surface area of S.

It also looks as if the approximation would improve if we refined the partition of R. In fact, the sums on the right-hand side of the above equation are approximation sums for the double integral

$$\iint_R \frac{1}{|\cos\gamma|}\, dA.$$

We therefore define the area of S to be the value of this integral whenever it exits.

How to find $\frac{1}{|\cos\gamma|}$?

For any surface $f(x, y, z) = c$, we have $|\nabla f.\text{p}| = |\nabla \text{f}|\ |\text{p}|\ |\cos\gamma|$, hence

$$\frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f.\text{p}|}.$$

The area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area } = \iint_R \frac{|\nabla f|}{|\nabla f.\text{p}|} dA,$$

where p is a unit vector normal to R and $\nabla f.\text{t} \neq 0$.

Surface Area

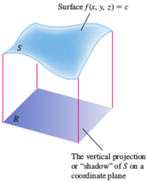
Thus, the area is the double integral over R of the magnitute of ∇f divided by the magnitude of the scalar component of ∇f normal to R.

We reached the above equation under the assumption that $\nabla f.\text{p} \neq 0$ throughout R and ∇f is continuous.

Whenever the integral exists, however, we define its value to be the area of the portion of the surface $f(x, y, z) = c$ that lies over R.

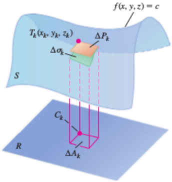
We now show how to integrate a function over a surface, using the ideas just developed for calculating surface area.

Suppose, for example, that we have an electrical charge distributed over a surface $f(x, y, x) = c$ like the one shown in figure and that the function $g(x, y, z)$ gives the charge per unit area (charge density) at each point on S.



Then we may calculate the total charge on S as an integral in the following way.

We partition the shadown region R on the ground plane beneath the surface into small rectangles of the kind we would use if we were defining the surface area of S. Then directly above each ΔA_k lies a patch a surface $\Delta\sigma_k$ that we approximate with a parallelogram-shaped portion of tangent plane, ΔP_k.



Up to this point the construction proceeds as in the definition of surface area, but we take one additional setp. We evaluate g at (x_k, y_k, z_k) and then approximate the total charge on the surface patch $\Delta\sigma_k$ by the product $g(x_k, y_k, z_k)\Delta P_k$.

The rationale is that when the partition of R is sufficiently fine, the value of g throughout $\Delta\sigma_k$ is nearly constant and ΔP_k is nearly the same as $\Delta\sigma_k$.

The total charge over S is then approximated by the sum

$$\text{Total charge } \approx \sum g(x_k, y_k, z_k)\Delta P_k = \sum g(x_k, y_k, z_k)\frac{\Delta A_k}{|\cos\gamma_k|}.$$

Surface Integrals

If f, the **function defining the surface S, and its partial derivatives are continuous, and if g is continuous over S,** then the sums on the right-hand side of above equation approach the limit

$$\iint_R g(x,y,z)\frac{dA}{|\cos\gamma|} = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f.\mathrm{p}|}dA$$

as the partition of R is refined in the usual way.

This limit is called the integral of g over the surface S and is calculated as a doube integral over R. The value of the integral is the total charge on the surface S.

Surface Integrals

The above formula defined the integral of any function g over the surface S as long as the integral exists.

If R is the shadow region of a surface S defined by the equation $f(x, y, z) = c$, and g is a continuous function defined at the points of S, then the integral of g over S is the integral

$$\iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f . \mathrm{p}|} dA,$$

where p is a unit vector normal to R and $\nabla f . \ \mathrm{p} \neq 0$. The integral itself is called a surface integral.

Surface Integrals

The integral

$$\iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f.\mathrm{p}|}\,dA$$

takes on different meaning in different applications.

- If g has the constant value 1, the integral gives the area of S.
- If g gives the mass density of a thin shell of material modeled by S, the integral gives the mass of the shell.

Algebraic Properties : The Surface Area Differential

We can abbreviate the integral

$$\iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f.\mathrm{p}|}dA$$

by writing $d\sigma$ for $(|\nabla f|/|\nabla f.\mathrm{p}|)dA$.

The surface area differential is $d\sigma = \frac{|\nabla f|}{|\nabla f.\mathrm{p}|}dA$. The differntial formula for surface integral is

$$\iint_S g\ d\sigma.$$

Surface Integrals

Surface integrals behave like other double integrals, the integral of the sum of two functions being the sum of their integrals and so on.

The domain additivity property takes the form

$$\iint_S g\ d\sigma = \iint_{S_1} g\ d\sigma \iint_{S_2} g\ d\sigma + \cdots + \iint_{S_n} g\ d\sigma.$$

The idea is that if S is partitioned by smooth curves into a finite number of non-overlapping smooth patches (i.e., if S is piecewise smooth), then the integral over S is the sum of the integrals over the patches.

Surface Integrals

Thus, the integral of a function over the surface of a cube is the sum of the integrals over the faces of the cube.

We integrate over a turtle shell of welded plates by integrating one plate at a time and adding the results.

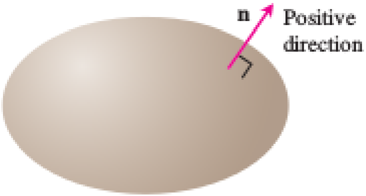
Orientation

We call a smooth surface S **orientable** or **two-sided** if it possible to define a field n of unit normal vectors on S that varies continuouly with position.

Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we choose n on a closed surface to point outward.

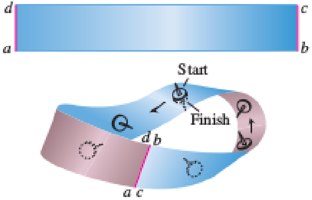
Orientation

Once n has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector n at any point is called the **positive direction** at that point.



Orientation

The Mobius band



is not orientable. Now mater where you start to construct a continuous unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out.

The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exits.

The Surface Integral for Flux

Suppose that F is a continuous vector field defined over an oriented surface S and that n is the chosen unit normal field on the surface.

We call the integral of F.n over S the flud across S in the positive direction. Thus, the flux is the integral over S of the scalar component of F in the direction of n.

The **flux** of a three-dimensional vector field F across an oriented surface S in the direction of n is given by the formula

$$\text{Flux} = \iint_S \mathrm{F.n} d\sigma.$$

The Surface Integral for Flux

The definition is analogous to the flux of a two-dimensional field F across a plane curve C. In the plane, the flux is

$$\int_C \mathrm{F}.\mathrm{n}ds,$$

the integral of the scalar component of F normal to the curve.

If F is the velocity field of a three-dimensional fluid flow, the flux of F across S is the net rate at which fluid is crossing S in the chosen positive direction.

The Surface Integral for Flux

If S is part of a level surface $g(x, y, z) = c$, then n may be taken to be one of the two fields

$$\mathsf{n} = \pm \frac{\nabla g}{|\nabla g|},$$

depending on which one gives the preferred direction. The corresponsing flux is

$$\begin{aligned} \text{Flux} &= \iint_S \mathsf{F}.\mathsf{n}\, d\sigma \\ &= \iint_S \left(\mathsf{F}.\frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g.\mathsf{p}|} dA \\ &= \iint_S \mathsf{F}.\frac{\pm \nabla g}{|\nabla g.\mathsf{p}|} dA \end{aligned}$$

Thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their moments and masses are calculated as follows.

Mass

Let $\delta(x, y, z)$ be the density at (x, y, z) mass per unit area.
$M = \iint_S \delta(x, y, z) d\sigma$

First moments above the coordinate planes :

$M_{yz} = \iint_S x \ \delta \ d\sigma, \quad M_{xz} = \iint_S y \ \delta \ d\sigma, \quad M_{xy} = \iint_S z \ \delta \ d\sigma$

Coordinates of center of mass :

$\overline{x} = M_{yz}/M, \quad \overline{y} = M_{xz}/M, \quad \overline{z} = M_{xy}/M.$

Moments of inertia

$$\begin{aligned} I_x &= \iint_S (y^2+z^2)\ \delta\ d\sigma \\ I_y &= \iint_S (x^2+z^2)\ \delta\ d\sigma \\ I_z &= \iint_S (x^2+y^2)\ \delta\ d\sigma. \end{aligned}$$

Let $r(x,y,z)$ be the distance from point (x,y,z) to line L.

$$I_L = \iint_S r^2\ \delta\ d\sigma.$$

Radius of gyration about a line L : $R_L = \sqrt{I_L/M}$.

Formulas for a Surface Integral

For a smooth surface S defined **parametrically** as

$$\mathrm{r}(u,v) = f(u,v)\mathrm{i} + g(u,v)\mathrm{j} + h(u,v)\mathrm{k}, \quad (u,v) \in R,$$

and a continuous function $G(x,y,z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x,y,z)\ d\sigma = \iint_R G(f(u,v), g(u,v), h(u,v))\ |\mathrm{r}_u \times \mathrm{r}_v|\ du\ dv.$$

Formulas for a Surface Integral

For a surface S given **implicitly** by

$$F(x, y, z) = c$$

where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\ d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F.\mathrm{p}|} dA,$$

where p is a unit vector normal to R and $\nabla F.\mathrm{p} \neq 0$.

Formulas for a Surface Integral

For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y))\, \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

Exercise 1.

Integrate $G(x, y, z) = y + z$ *over the surface of the wedge in the first octant bounded by the coordinate planes and the planes* $x = 2$ *and* $y + z = 1$.

Solution for Exercise 1

On the face S in the xz-plane, we have $y = 0 \Rightarrow f(x, y, z) = y = 0$ and $G(x, y, z) = G(x, 0, z) = z \Rightarrow$ **p=j** and $\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dxdz \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S (y + z) d\sigma = \int_0^1 \int_0^2 z dxdz = \int_0^1 2z dz = 1.$

On the face in the xy-plane, we have $z = 0 \Rightarrow f(x, y, z) = z = 0$ and $G(x, y, z) = G(x, y, 0) = y \Rightarrow$ **p=k** and $\nabla f = \mathbf{k}$ $\Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dxdy \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S y d\sigma = \int_0^1 \int_0^2 y dxdy = 1.$

On the triangular face in the plane $x = 2$ we have $f(x, y, z) = x = 2$ and $G(x, y, z) = G(2, y, z) = y + z \Rightarrow$ **p=i** and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dzdy \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S (y + z) d\sigma = \int_0^1 \int_0^{1-y} (y + z) dzdy = \int_0^1 \frac{1}{2}(1 - y^2) dy = \frac{1}{3}.$

On the triangular face in the yz-plane, we have $x = 0 \Rightarrow f(x, y, z) = x = 0$ and $G(x, y, z) = G(0, y, z) = y + z$ $\Rightarrow$ **p=i** and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dzdy \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S (y + z) d\sigma = \int_0^1 \int_0^{1-y} (y + z) dzdy = \frac{1}{3}$. Finally, on the sloped face, we have $y + z = 1 \Rightarrow f(x, y, z) = y + z = 1$ and $G(x, y, z) = y + z = 1 \Rightarrow$ **p=k** and $\nabla f = \mathbf{j}+\mathbf{k} \Rightarrow |\nabla f| = \sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{2} dxdy \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S (y + z) d\sigma = \int_0^1 \int_0^2 \sqrt{2} dxdy = 2\sqrt{2}$. Therefore,

$$\iint\limits_{wedge} G(x, y, z) d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}.$$

Exercise 2.

Integrate $G(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a$, $y = b$, and $z = c$.

Solution for Exercise 2

On the faces in the coordinate planes, $G(x, y, z) = 0 \Rightarrow$ the integral over these faces is 0.
On the face $x = a$, we have $f(x, y, z) = x = a$ and $G(x, y, z) = G(a, y, z) = ayz \Rightarrow$ **p=i** and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and

$|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dydz \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S ayz d\sigma = \int_0^c \int_0^b ayz dydz = \frac{ab^2c^2}{4}$.

On the face $y = b$, we have $f(x, y, z) = y = b$ and $G(x, y, z) = G(x, b, z) = bxz \Rightarrow$ **p=j** and $\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$ and

$|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dxdz \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S bxz d\sigma = \int_0^c \int_0^a bxz dxdz = \frac{a^2bc^2}{4}$.

On the face $z = c$, we have $f(x, y, z) = z = c$ and $G(x, y, z) = G(x, y, c) = cxy \Rightarrow$ **p=k** and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and

$|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dydx \Rightarrow \iint\limits_S G d\sigma = \iint\limits_S cxy d\sigma = \int_0^b \int_0^a cxy dxdy = \frac{a^2b^2c}{4}$. Therefore,

$\iint\limits_S G(x, y, z) d\sigma = \frac{abc(ab+ac+bc)}{4}$.

Exercise 3.

1. *Integrate* $G(x, y, z) = x\sqrt{y^2 + 4}$ *over the surface cut from the parabolic cylinder* $y^2 + 4z = 16$ *by the planes* $x = 0$, $x = 1$, *and* $z = 0$.
2. *Integrate* $G(x, y, z) = z - x$ *over the portion of the graph of* $z = x + y^2$ *above the triangle in the* xy*-plane having vertices* $(0, 0, 0)$, $(1, 1, 0)$, *and* $(0, 1, 0)$. *(See accompanying figure.)*
3. *Integrate* $G(x, y, z) = xyz$ *over the triangular surface with vertices* $(1, 0, 0)$, $(0, 2, 0)$, *and* $(0, 1, 1)$.
4. *Integrate* $G(x, y, z) = x - y - z$ *over the portion of the plane* $x + y = 1$ *in the first octant between* $z = 0$ *and* $z = 1$ *(see the accompanying figure).*

Solution for Exercise 3

1. $f(x,y,z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+16} = 2\sqrt{y^2+4}$ and $\mathbf{p}$=$\mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$
$\Rightarrow d\sigma = \frac{2\sqrt{y^2+4}}{4} dxdy \Rightarrow \iint\limits_S G d\sigma = \int_{-4}^{4}\int_0^1 \left(x\sqrt{y^2+4}\right)\left(\frac{\sqrt{y^2+4}}{2}\right) dxdy =$
$\int_{-4}^{4}\int_0^1 \frac{x(y^2+4)}{2} dxdy = \int_{-4}^{4} \frac{1}{4}(y^2+4)dy = \frac{1}{2}\left[\frac{y^3}{3}+4y\right]_0^4 = \frac{1}{2}\left(\frac{64}{3}+16\right) = \frac{56}{3}$
2. $f(x,y,z) = x + y^2 - z = 0 \Rightarrow \nabla f = \mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2+2} = \sqrt{2}\sqrt{2y^2+1}$ and $\mathbf{p}$=$\mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2y^2+1}}{1} dxdy \Rightarrow \iint\limits_S G d\sigma =$
$\int_0^1\int_0^y (x + y^2 - x)\sqrt{2}\sqrt{2y^2+1} dxdy = \sqrt{2}\int_0^1\int_0^y y^2\sqrt{2y^2+1} dxdy$
$= \sqrt{2}\int_0^1 y^3\sqrt{2y^2+1} dy = \frac{6\sqrt{6}+\sqrt{2}}{30}$
3. $f(x,y,z) = 2x + y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{6}$ and $\mathbf{p}$=$\mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{6}}{1} dydx$
$\Rightarrow \iint\limits_S G d\sigma = \int_0^1\int_{1-x}^{2-2x} xy(2-2x-y)\sqrt{6} dydx = \sqrt{6}\int_0^1\int_{1-x}^{2-2x} (2xy - 2x^2y - xy^2) dydx$
$= \sqrt{6}\int_0^1 (\frac{2}{3}x - 2x^2 + 2x^3 - \frac{2}{3}x^4) dx = \frac{\sqrt{6}}{30}$
4. $f(x,y,z) = x + y = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j} \Rightarrow |\nabla f| = \sqrt{2}$ and $\mathbf{p}$=$\mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{2}}{1} dzdx \Rightarrow \iint\limits_S G d\sigma = \int_0^1\int_0^1 (x - (1-x) - z)\sqrt{2} dzdx = \sqrt{2}\int_0^1\int_0^1 (2x - z - 1) dzdx$
$= \sqrt{2}\int_0^1 (2x - \frac{3}{2}) dx = -\frac{\sqrt{2}}{2}$.

Exercise 4.

Find the flux of the field F *across the portion of the given surface in the specified direction.*

$$\mathsf{F}(x, y, z) = yx^2\mathsf{i} - 2\mathsf{j} + xz\mathsf{k}$$

and S : *rectangular surface* $y = 0, \quad -1 \leq x \leq 2, \quad 2 \leq z \leq 7$, *direction* $-\mathsf{j}$

Solution for Exercise 4 :

$g(x, y, z) = y$, $\mathbf{p} = -\mathbf{j} \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow$ Flux $= \iint\limits_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint\limits_R (\mathbf{F} \cdot -\mathbf{j}) dA = \int_{-1}^{2} \int_{2}^{7} 2dzdx = \int_{-1}^{2} 2(7-2)dx = 10(2+1) = 30.$

Exercise 5.

In the following exercises, find the flux of the field F *across the portion of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant in the direction away from the origin.*

1. $\mathrm{F}(x, y, z) = -y\mathrm{i} + x\mathrm{j}$
2. $\mathrm{F}(x, y, z) = zx\mathrm{i} + zy\mathrm{j} + z^2\mathrm{k}$
3. $\mathrm{F}(x, y, z) = \frac{x\mathrm{i}+y\mathrm{j}+z\mathrm{k}}{\sqrt{x^2+y^2+z^2}}$

Solution for Exercise 5

1. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a;$
$\mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-xy}{a} + \frac{xy}{a}$
$= 0; |\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z} dA \Rightarrow \text{Flux} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint\limits_S 0\, d\sigma = 0$

2. $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z\left(\frac{x^2+y^2+z^2}{a}\right) = az$
$\Rightarrow \text{Flux} = \iint\limits_R (za)(\frac{a}{z}) dxdy = \iint\limits_R a^2 dxdy = a^2(\text{Area of } R) = \frac{1}{4}\pi a^4$

3. $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{(\frac{x^2}{a})+(\frac{y^2}{a})+(\frac{z^2}{a})}{\sqrt{x^2+y^2+z^2}} = \frac{(\frac{a^2}{a})}{a} = 1$
$\Rightarrow \text{Flux} = \iint\limits_R \frac{a}{z} dxdy = \iint\limits_R \frac{a}{\sqrt{a^2-(x^2+y^2)}} dxdy = \int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2-r^2}} r\,dr\,d\theta = \frac{\pi a^2}{2}$

Exercise 6.

1. *Find the flux of the field* $\mathrm{F}(x, y, z) = 4x\mathrm{i} + 4y\mathrm{j} + 2\mathrm{k}$ *outward (away from the z-axis) through the surface cut from the bottom of the paraboloid* $z = x^2 + y^2$ *by the plane* $z = 1$*.*
2. *Let* S *be the portion of the cylinder* $y = e^x$ *in the first octant that projects parallel to the* x*-axis onto the rectangle* $R_{yz} : 1 \leq y \leq 2$*,* $0 \leq z \leq 1$ *in the yz-plane (see the accompanying figure). Let* n *be the unit vector normal to* S *that points away from the yz-plane. Find the flux of the field* $\mathrm{F}(x, y, z) = -2\mathrm{i} + 2y\mathrm{j} + z\mathrm{k}$ *across* S *in the direction of* n*.*

Solution for Exercise 6

1. $g(x,y,z) = x^2 + y^2 - z = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2+y^2)+1}$
$\Rightarrow \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4(x^2+y^2)+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}$;
$\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4(x^2+y^2)+1}\,dA$
$\Rightarrow \text{Flux} = \iint\limits_R \left(\frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}\right)\sqrt{4(x^2+y^2)+1}\,dA = \iint\limits_R (8x^2+8y^2-2)\,dA$; $z = 1$ and $x^2+y^2 = z$
$\Rightarrow x^2+y^2 = 1 \Rightarrow \text{Flux} = \int_0^{2\pi}\int_0^1 (8r^2-2)\,r\,dr\,d\theta = 2\pi$
2. $g(x,y,z) = y - e^x = 0 \Rightarrow \nabla g = -e^x\mathbf{i} + \mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x}+1} \Rightarrow \mathbf{n} = \frac{e^x\mathbf{i}-\mathbf{j}}{\sqrt{e^{2x}+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-2e^x-2y}{\sqrt{e^{2x}+1}}$; $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g\cdot\mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x}+1}}{e^x}\,dA$
$\Rightarrow \text{Flux} = \iint\limits_R \left(\frac{-2e^x-2y}{\sqrt{e^{2x}+1}}\right)\left(\frac{\sqrt{e^{2x}+1}}{e^x}\right)dA = \iint\limits_R \frac{-2e^x-2e^x}{e^x}\,dA$
$= \iint\limits_R -4\,dA = \int_0^1\int_1^2 -4\,dy\,dz = -4$

Exercise 7.

Find the outward flux of the field $\mathsf{F} = 2xy\mathsf{i} + 2yz\mathsf{j} + 2xz\mathsf{k}$ *across the surface of the cube cut from the first octant by the planes* $x = a$, $y = a$, $z = a$

Solution for Exercise 7

On the face $z = a : g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2xz = 2ax$ since $z = a$; $d\sigma = dxdy \Rightarrow$ Flux $= \iint\limits_R 2axdxdy = \int_0^a \int_0^a 2axdxdy = a^4$.

On the face $z = 0 : g(x, y, z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2xz = 0$ since $z = 0$; $d\sigma = dxdy \Rightarrow$ Flux $= \iint\limits_R 0dxdy = 0$.

On the face $x = a : g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2xy = 2ay$ since $x = a$; $d\sigma = dydz \Rightarrow$ Flux $\int_0^a \int_0^a 2aydydz = a^4$.

On the face $x = 0 : g(x, y, z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2xy = 0$ since $x = 0 \Rightarrow$ Flux $= 0$.

On the face $y = a : g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1; \mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} = 2yz = 2az$ since $y = a$; $d\sigma = dzdx \Rightarrow$ Flux $= \int_0^a \int_0^a 2azdzdx = a^4$.

On the face $y = 0 : g(x, y, z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1; \mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -2yz = 0$ since $y = 0; \Rightarrow$ Flux $= 0$. Therefore, Total Flux $= 3a^4$.

Exercise 8.

1. Centroid : *Find the centroid of the portion of the sphere* $x^2 + y^2 + z^2 = a^2$ *that lies in the first octant.*
2. Thin shell of constant density : *Find the center of mass and the moment of inertia about the z-axis of a thin shell of constant density* δ *cut from the cone* $x^2 + y^2 - z^2 = 0$ *by the planes* $z = 1$ *and* $z = 2$.
3. Conical surface of constant density : *Find the moment of inertia about the z-axis of a thin shell of constant density* δ *cut from the cone* $4x^2 + 4y^2 - z^2 = 0$, $z \geq 0$, *by the circular cylinder* $x^2 + y^2 = 2x$ *(see the accompanying figure).*

Solution for Exercise 8

1. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+4z^2} = 2a$; $\mathbf{p}$=$\mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \geq 0 \Rightarrow d\sigma = \frac{2a}{2z}dA = \frac{a}{z}dA$; $M = \iint_S \delta \, d\sigma = \frac{\delta}{8}$(surface area of sphere)$=\frac{\delta\pi a^2}{2}$; $M_{xy} = \iint_S z\delta\, d\sigma = \delta \iint_R z\left(\frac{a}{z}\right)dA = a\delta \iint_R dA = a\delta \int_0^{\pi/2}\int_0^a r\,dr\,d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\delta\pi a^3}{4}\right)\left(\frac{2}{\delta\pi a^2}\right) = \frac{a}{2}$. Because of symmetry, $\bar{x} = \bar{y} = \frac{a}{2} \Rightarrow$ the centroid is $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.
2. Because of symmetry, $\bar{x} = \bar{y} = 0$; $M = \iint_S \delta\, d\sigma = \delta \iint_S d\sigma =$(Area of S)$= 3\pi\sqrt{2}\delta$;
   $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2}$; $\mathbf{p}$=$\mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2+y^2+z^2}}{2z}dA = \frac{\sqrt{x^2+y^2+(x^2+y^2)}}{z}dA = \frac{\sqrt{2}\sqrt{x^2+y^2}}{z}dA \Rightarrow M_{xy} = \delta \iint_R z\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right)dA = \delta \iint_R \sqrt{2}\sqrt{x^2+y^2}\,dA = \delta \int_0^{2\pi}\int_1^2 \sqrt{2}r^2\,dr\,d\theta = \frac{14\pi\sqrt{2}}{3}\delta \Rightarrow \bar{z} = \frac{\left(\frac{14\pi\sqrt{2}}{3}\delta\right)}{3\pi\sqrt{2}\delta} = \frac{14}{9} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right)$.
   Next,
   $I_z = \iint_S (x^2+y^2)\delta\, d\sigma = \iint_R (x^2+y^2)\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right)\delta\, dA = \delta\sqrt{2}\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta = \frac{15\pi\sqrt{2}}{2}\delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}$
3. $f(x,y,z) = 4x^2+4y^2-z^2 = 0 \Rightarrow \nabla f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2+64y^2+4z^2} = 2\sqrt{16x^2+16y^2+z^2} = 2\sqrt{4z^2+z^2} = 2\sqrt{5}z$ since $z \geq 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}z}{2z}dA = \sqrt{5}dA \Rightarrow I_z = \iint_S (x^2+y^2)\delta\, d\sigma = \delta\sqrt{5}\iint_R (x^2+y^2)\,dx\,dy = \delta\sqrt{5}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \frac{3\sqrt{5}\pi\delta}{2}$

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